

## Polarized maser growth

D. B. Melrose and A. C. Judge

*School of Physics, University of Sydney, Sydney, NSW 2006, Australia*

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A polarized maser is assumed to operate in an anisotropic medium with natural modes polarized differently to the maser. It is shown that when the spatial growth rate and the generalized Faraday rotation rate are comparable, the polarization of the growing radiation is different from those of the maser and medium. In particular, for a linearly polarized maser operating in a medium with linearly polarized natural modes, the growing radiation is partially circularly polarized. This provides a previously unrecognized source of circular polarization that may be relevant to pulsar radio emission.

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### I. INTRODUCTION

When a maser operates in an anisotropic medium, the natural polarization of the maser need not correspond to the natural polarization of the wave modes in the medium. For example, consider an idealized molecular line maser that is Zeeman split into transitions that are intrinsically polarized, two with opposite circular polarizations and one with linear polarization. Suppose such a linearly polarized maser operates in an ionized plasma whose natural modes are nearly circularly polarized. One expects the polarization of the amplifying radiation to be characteristic of the maser provided that the growth rate is much greater than the Faraday rotation rate and of the faster growing of the two natural modes when the Faraday rotation rate is much larger than the growth rate. However, it is not clear what the polarization is in the intermediate case where the two rates are comparable. Another example is relevant to pulsar radio emission. In this case, the natural wave modes of the highly relativistic pulsar plasma are nearly linearly polarized in most cases. The suggested pulsar radio emission mechanisms include at least two that involve masers with polarizations that differ from the polarizations of the natural modes: maser curvature emission [1,2] and anomalous cyclotron maser emission [3,4]. For maser curvature emission the maser favors a linear polarization that is different from the linear polarization of the natural modes, provided that the particles responsible for the emission are independent of those determining the dispersion of these modes. Here we point out that in the intermediate case, the polarization is elliptical, providing a previously unrecognized source of circular polarization.

The generic problem discussed in this paper applies to any polarized maser operating in any anisotropic medium in which the modes are transverse to a first approximation (the “weak anisotropy” approximation). Let  $-\mu$  be the polarized growth rate and let  $r$  be the generalized Faraday rotation rate. Generalized Faraday rotation is most easily visualized on the Poincaré sphere: regard the diagonal (denoted by the direction  $\hat{r}$  here) defined by the polarizations of the two natural modes of the medium as an axis; then, the polarization point, defined by the Stokes parameters for any given radiation, rotates about this axis at the rate  $r$  due to components in the two modes getting out of phase. The generic problem is characterized by two numbers:  $\mu/r$  and the cosine  $g$  of the angle

between the diagonals on the Poincaré sphere representing the polarizations of the maser and natural modes. The parameter  $g$  is related to the projection between the polarization vectors (in coordinate space) of the maser and one mode of the medium: if both are linear,  $g$  is the cosine of twice the angle between them. In particular, linear polarizations that are orthogonal on the Poincaré sphere correspond to polarization vectors at  $45^\circ$  (modulo  $90^\circ$ ) to each other.

The transfer equation for polarized radiation in the weak anisotropy approximation is written down, applied to a polarized maser, and formally solved in Sec. II. The growth of polarized radiation is discussed in Sec III, starting with the special cases where the polarization of maser and medium are parallel and perpendicular on the Poincaré sphere and then with illustrative examples of the growth in the general case, emphasizing the asymptotic solution after many growth lengths. Some implications of the results are discussed briefly in Sec. IV.

### II. FORMAL TREATMENT OF POLARIZED MASER GROWTH

The transfer equation in the Mueller calculus is used to treat the transfer of radiation in the weak anisotropy approximation. The relevant equation was written down in [5] and discussed further in [6–8]; cf. also the monographs [9–11].

#### A. Transfer equation in the weak anisotropy approximation

For radiation propagating along the  $z$  direction, the transfer equation is

$$dS_A/dz = \alpha_A + (-\mu_{AB} + r_{AB})S_B, \quad (1)$$

where  $S_A$  with  $A=I, Q, U, V$  is the Stokes vector,  $\alpha_A$  are emission coefficients,  $\mu_{AB}$  are absorption coefficients, and  $r_{AB}$  are generalized Faraday rotation rates. These have the forms

$$S_A = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}, \quad \alpha_A = \begin{pmatrix} \alpha_I \\ \alpha_Q \\ \alpha_U \\ \alpha_V \end{pmatrix}, \quad \mu_{AB} = \begin{pmatrix} \mu_I & \mu_Q & \mu_U & \mu_V \\ \mu_Q & \mu_I & 0 & 0 \\ \mu_U & 0 & \mu_I & 0 \\ \mu_V & 0 & 0 & \mu_I \end{pmatrix},$$

$$r_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r_V & r_U \\ 0 & r_V & 0 & -r_Q \\ 0 & -r_U & r_Q & 0 \end{pmatrix}. \quad (2)$$

An alternative way of writing Eq. (1) that is sometimes convenient is in terms of unpolarized components and 3-vectors describing the polarized components. Writing  $\mathbf{S} = (Q, U, V)$ ,  $\boldsymbol{\alpha} = (\alpha_Q, \alpha_U, \alpha_V)$ ,  $\mathbf{r} = (r_Q, r_U, r_V)$ , and  $\boldsymbol{\mu} = (\mu_Q, \mu_U, \mu_V)$ , Eq. (1) with Eqs. (2) becomes

$$dI/dz = \alpha_I - \mu_I I - \boldsymbol{\mu} \cdot \mathbf{S}, \quad d\mathbf{S}/dz = \boldsymbol{\alpha} - \mu_I \mathbf{S} - \boldsymbol{\mu} I + \mathbf{r} \times \mathbf{S}. \quad (3)$$

There are two general ways of solving Eq. (1). The first method involves regarding Eq. (1) as a matrix equation and solving it by finding the eigenvalues and eigenfunctions. Integration of the eigenvalue equation is elementary. This method is used here. The other method involves integrating the matrix equation (1) directly; cf. Appendix A.

Note that the theory is invariant under cyclic permutations of  $Q, U, V$ . Thus the most general case may be treated by considering the special case in which only  $\mu_V, r_Q$ , and  $r_V$  are nonzero: any other particular case then follows by appropriate relabeling.

### B. Completely polarized maser

Two simplifications are made for convenience before solving Eqs. (2): the terms  $\alpha_A$  and  $\mu_I$  are omitted. The neglect of  $\alpha_A$  corresponds to neglecting spontaneous emission. Physically, this is justified if amplified background noise is more important in the output of the maser than amplified spontaneous emission. It is straightforward to include  $\alpha_A \neq 0$  and to treat amplified spontaneous emission, and the conclusions of this paper are unaffected by including amplified spontaneous emission. The neglect of  $\mu_I$  involves no actual loss of generality provided that one reinterprets  $S_A$  in Eq. (1) as  $S'_A = \exp(\mu_I z) S_A$  in a modified Eqs. (1) and (2) with  $\mu_I$  omitted in the matrix  $\mu_{AB}$ . It follows that the polarization of the radiation, which is determined by the ratios of the Stokes parameters, is unaffected by  $\mu_I$  and that one is justified in neglecting  $\mu_I$  when considering the polarization.

It is helpful to introduce the concept of a completely polarized maser, specifically one with  $\mu_I = 0$ . This concept is useful for isolating the polarization characteristics of the growing radiation. Although there is no physical reason why such an idealized maser should not exist, the conditions under which it would occur are somewhat contrived. The assumption  $\mu_I = 0$  has no effect on the results of this paper relating to the relative amplitudes of the Stokes parameters in the growing radiation.

### C. General and special cases

In the general case, the polarization of the maser and the polarization of the medium are different. It is helpful to identify the polarizations on the Poincaré sphere. A specific polarization is represented by a point on the sphere, with circu-

lar polarizations represented by the poles and linear polarizations by points around the equator. Orthogonal polarizations are represented by points on the opposite side of a diagonal that passes through the center of the sphere. A completely polarized maser defines one diagonal through the sphere, and polarized growth alone causes radiation with one of the two orthogonal polarizations to grow and the other to damp. The polarization of the natural modes defines another diagonal through the sphere. Generalized Faraday rotation alone is described by the term  $d\mathbf{S}/dz = \mathbf{r} \times \mathbf{S}$  in Eqs. (3), and this corresponds to the polarization point rotating about the diagonal at a constant latitude relative to it. The diagonals representing the polarizations of the maser and of the medium correspond to the directions  $-\boldsymbol{\mu}$  and  $\mathbf{r}$ , respectively, through the center of the Poincaré sphere. In the general case these are neither parallel nor perpendicular to each other. The interplay between polarized growth and generalized Faraday rotation depends only on the ratio of the lengths  $\mu, r$ , which define the growth rate to the Faraday rotation rate, and on the angle between  $\boldsymbol{\mu}$  and  $\mathbf{r}$ .

To treat the general case, it suffices to consider the special case  $\mu_V \neq 0, \mu_Q = \mu_U = 0, r_V, r_Q \neq 0$ , and  $r_U = 0$  and to appeal to the symmetry of the theory. In the 3-vector formalism used in Eqs. (3) this choice corresponds to orienting the axes on the Poincaré sphere such that the  $z$  axis is along  $\boldsymbol{\mu}$  and  $\mathbf{r}$  is in the  $x$ - $z$  plane. The general case is obtained by a rotation of the Poincaré sphere. On writing

$$Q/I = \cos(2\chi)\cos(2\psi), \quad U/I = \cos(2\chi)\sin(2\psi), \quad V/I = \sin(2\chi), \quad (4)$$

with analogous angles introduced for  $\boldsymbol{\alpha}, \boldsymbol{\mu}$ , and  $\mathbf{r}$ , a rotation on the Poincaré sphere is equivalent to a conventional rotation from an initial pair of polar and azimuthal angles  $\chi, \psi$  to a new pair  $\chi', \psi'$ , say. The relation (4) for the primed variables then determines  $Q', U', V'$  in terms of  $Q, U, V$  and similarly for the angles corresponding to  $\boldsymbol{\alpha}, \boldsymbol{\mu}, \mathbf{r}$ .

### D. Characteristic equation

The transfer equation (1) may be solved by finding the eigenvalues and eigenfunctions of the matrix  $s_{AB} = -\mu_{AB} + r_{AB}$ , where now we assume  $\mu_I = 0$  and ignore spontaneous emission. Let an eigenvalue be denoted by  $\lambda$ . The characteristic equation is

$$\Lambda(\lambda) = \det(-\mu_{AB} + r_{AB} - \lambda \delta_{AB}) = \lambda^4 - \lambda^2 \lambda_0^2 - \mu^2 r^2 g^2 = 0, \quad (5)$$

where the two invariants in the problem are

$$\lambda_0^2 = \mu^2 - r^2, \quad g = \boldsymbol{\mu} \cdot \mathbf{r} / \mu r. \quad (6)$$

For  $g \neq 0$ , Eq. (5) has two real and two imaginary solutions for  $\lambda$ , and for  $g = 0$ , Eq. (5) has a double solution  $\lambda = 0$  and either two real (for  $\lambda_0^2 > 0$ ) or two imaginary (for  $\lambda_0^2 < 0$ ) solutions.

The eigenvalues are found by constructing the matrix of cofactors,  $\Lambda_{AB}(\lambda)$ , so that one has

$$\Lambda_{AB}(\lambda)(-\mu_{BC} + r_{BC} - \lambda \delta_{BC}) = \Lambda(\lambda) \delta_{AC}. \quad (7)$$

Premultiplying Eq. (1) by the matrix of cofactors then gives

$$\frac{d}{dz}[\Lambda_{AB}(\lambda)S_B] = \Lambda(\lambda)S_A + \lambda\Lambda_{AB}(\lambda)S_B. \quad (8)$$

When the condition (5) is satisfied,  $\Lambda_{AB}(\lambda)S_B$  is an eigenfunction.

Let  $\lambda_i$ ,  $i=1-4$ , be the four eigenvalues. Then the four eigenfunctions are given by  $\Lambda_{AB}(\lambda_i)S_B$  for any  $A$  or for an arbitrary linear combination of these functions. The eigenfunctions constructed for different  $A$  are proportional to each other, in the sense that the ratios of the coefficients of  $I, Q, U, V$  are the same for different  $A$  for any given  $\lambda_i$ .

### E. Eigenvalues and eigenfunctions

In the general case, the two solutions of the eigenvalue equation (5) for  $\lambda^2$  are

$$\lambda^2 = \lambda_{\pm}^2, \quad \lambda_{\pm}^2 = \frac{1}{2}(\mu^2 - r^2) \pm \frac{1}{2}[(\mu^2 - r^2)^2 + 4(\boldsymbol{\mu} \cdot \mathbf{r})^2]^{1/2}. \quad (9)$$

With  $\lambda_+^2 > 0$  and  $\lambda_-^2 < 0$  both real, the four eigenvalues may be written

$$\lambda_1 = \lambda_+, \quad \lambda_2 = -\lambda_+, \quad \lambda_3 = i|\lambda_-|, \quad \lambda_4 = -i|\lambda_-|. \quad (10)$$

Parallel polarizations ( $\boldsymbol{\mu} \cdot \mathbf{r} = \mu r$ ) correspond to  $\lambda_+ \rightarrow \mu$ ,  $|\lambda_-| \rightarrow r$ , and perpendicular polarizations ( $\boldsymbol{\mu} \cdot \mathbf{r} = 0$ ) correspond to  $\lambda_+ \rightarrow (\mu^2 - r^2)^{1/2}$ ,  $|\lambda_-| \rightarrow 0$  for  $\mu^2 > r^2$  and to  $\lambda_+ \rightarrow 0$ ,  $|\lambda_-| \rightarrow (r^2 - \mu^2)^{1/2}$  for  $r^2 < \mu^2$ .

The eigenfunctions follow from Eq. (8) and may be identified by choosing any  $A$  in  $\Lambda_{AB}(\lambda_i)S_B$ . Choosing  $A=I$ , the (unnormalized) eigenfunctions are denoted  $S_i$  with  $i=1-4$ . It is convenient to introduce the unit vectors  $\hat{\mathbf{r}} = \mathbf{r}/r$ ,  $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}/\mu$ , and to choose as a set of three orthonormal vectors  $\hat{\mathbf{r}}, \hat{\boldsymbol{\mu}} \times \hat{\mathbf{r}}, \hat{\mathbf{r}} \times (\hat{\boldsymbol{\mu}} \times \hat{\mathbf{r}})$ , and to write

$$S_r = \hat{\mathbf{r}} \cdot \mathbf{S}, \quad S_{\perp} = \hat{\boldsymbol{\mu}} \times \hat{\mathbf{r}} \cdot \mathbf{S}, \quad S_{\mu} = \hat{\mathbf{r}} \times (\hat{\boldsymbol{\mu}} \times \hat{\mathbf{r}}) \cdot \mathbf{S}. \quad (11)$$

In terms of these quantities, the four eigenfunctions become

$$S_i = -\lambda_i(\lambda_i^2 + r^2)I + \mu g(\lambda_i^2 + r^2)S_r + \lambda_i \mu r S_{\perp} + \mu \lambda_i^2 S_{\mu}, \quad (12)$$

with  $g = \boldsymbol{\mu} \cdot \mathbf{r} / \mu r$ . The transfer equation reduces to  $dS_i/dz = \lambda_i S_i$ , and the solution is

$$S_i(z) = e^{\lambda_i z} S_i(0). \quad (13)$$

These four equations, Eq. (13) with  $i=1-4$ , may be rewritten in the form  $S_A(z) = M_{AB}(z)S_B(0)$  to identify the Mueller matrix  $M_{AB}(z)$ , but we do not do so explicitly here. However, note that the solution corresponds to two linear combinations of Stokes parameters varying as hyperbolic functions of  $\lambda_+ z$ , and two linear combinations varying as trigonometric functions of  $|\lambda_-|z$ ; each Stokes parameter at  $z$  is related to the Stokes parameters at  $z=0$  through a combination of these hyperbolic and trigonometric functions.

### F. Mueller matrix

A formal solution of Eq. (1), when spontaneous emission is ignored, is

$$S_A(z) = M_{AB}(z)S_B(0), \quad M_{AB}(z) = \exp[(-\mu_{AB} + r_{AB})z], \quad (14)$$

where  $M_{AB}(z)$  is the Mueller matrix. Explicit evaluation of the Mueller matrix directly is discussed in Appendix A. The evaluation is straightforward in the special cases of parallel and perpendicular polarizations, but not in the general case. The Mueller matrix in the general case may be constructed by solving Eq. (13) for the Stokes parameters. This is carried out in Appendix B.

## III. GROWTH OF POLARIZED RADIATION

The general solution of Eq. (1) is given by Eq. (13). It is helpful to consider two special cases before discussing the general case. The special cases are where the polarizations of the maser and medium are either parallel ( $g=1$ ) or perpendicular ( $g=0$ ) on the Poincaré sphere.

### A. Parallel polarization

Parallel polarizations define a single direction  $\hat{\mathbf{r}} = \hat{\boldsymbol{\mu}}$  on the Poincaré sphere. The eigenvalues (5) are  $\lambda = \pm\mu$  and  $\lambda = \pm ir$ , and the eigenfunctions are  $I \pm S_r$  and the two components of  $\hat{\mathbf{r}} \times \mathbf{S}$ . The solutions for the components that grow and damp are

$$I(z) = I(0)\cosh(\mu z) + S_r(0)\sinh(\mu z),$$

$$\hat{\mathbf{r}} \cdot \mathbf{S}(z) = I(0)\sinh(\mu z) + S_r(0)\cosh(\mu z). \quad (15)$$

In particular, radiation polarized in the same sense as the maser grows exponentially, and radiation polarized in the opposite sense damps exponentially. Generalized Faraday rotation of the other two components may be described by

$$\hat{\mathbf{r}} \times \mathbf{S}(z) = \hat{\mathbf{r}} \times \mathbf{S}(0)\cos(rz) + \hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \mathbf{S}(0)]\sin(rz). \quad (16)$$

For the particular case of circularly polarized maser and modes, Eq. (16) implies  $Q(z) = Q(0)\cos(r_V z) - U(0)\sin(r_V z)$ ,  $U(z) = U(0)\cos(r_V z) + Q(0)\sin(r_V z)$ , which corresponds to Faraday rotation of the plane of linear polarization,  $\psi(z) = \psi(0) + \frac{1}{2}r_V z$ ; cf. Eqs. (4).

### B. Perpendicular polarizations

The case where the polarization of the maser is perpendicular (on the Poincaré sphere) to the polarization of the natural modes corresponds to  $\boldsymbol{\mu} \cdot \mathbf{r} = 0$ . Then two of the eigenvalues (9) are null ( $\lambda=0$ ) and the other two are  $\lambda = \pm\lambda_0$ ,  $\lambda_0^2 = \mu^2 - r^2$ . These two eigenvalues are real for  $\mu^2 > r^2$  and imaginary for  $\mu^2 < r^2$ . The eigenfunctions with null eigenvalues are  $S_r$  and  $rI - \mu S_{\perp}$ , and the other two eigenfunctions are given by Eq. (12) with  $g=0$ . Solving in the case  $\lambda_0^2 > 0$  for given initial conditions ( $z=0$ ) gives  $S_r(z) = S_r(0)$  and

$$\begin{pmatrix} I(z) \\ S_\mu(z) \\ S_\perp(z) \end{pmatrix} = \begin{pmatrix} I(0) \\ S_\mu(0) \\ S_\perp(0) \end{pmatrix} + \begin{pmatrix} -\mu S_\mu(0) \\ -\mu I(0) + r S_\perp(0) \\ -r S_\mu(0) \end{pmatrix} \frac{\sinh(\lambda_0 z)}{\lambda_0} \\ + \begin{pmatrix} \mu^2 I(0) - \mu r S_\perp(0) \\ (\mu^2 - r^2) S_\mu(0) \\ \mu r I(0) - r^2 S_\perp(0) \end{pmatrix} \frac{\cosh(\lambda_0 z) - 1}{\lambda_0^2}. \quad (17)$$

For  $\lambda_0^2 < 0$  the hyperbolic functions are replaced by trigonometric functions in Eq. (17).

Two interesting features appear for perpendicular polarizations. First, exponential growth occurs only for  $\mu^2 > r^2$ . For  $\mu^2 < r^2$  the system is periodic, but unlike generalized Faraday rotation, the oscillations involve the intensity  $I$ . For  $\mu^2 \rightarrow r^2$  the oscillation rate goes to zero and the amplitude of the oscillations becomes arbitrarily large. When the two rates are equal,  $\mu^2 = r^2$ , power-law growth occurs; cf. Eq. (A8) in Appendix A. Second, the polarization of the growing radiation has a component  $S_\perp$  that corresponds to the polarization of neither the maser nor the natural modes. Retaining only the exponentially growing terms, Eq. (17) implies that after many growth lengths, the ratio of the Stokes parameters approaches

$$I:S_r:S_\perp:S_\mu = 1:0:r/\mu:-\sqrt{\mu^2 - r^2}/\mu. \quad (18)$$

[Note that  $S_r(z) = S_r(0)$  remains constant at its initial value, and hence its ratio to  $I$  tends to zero after many growth lengths.] As the ratio  $\mu^2/r^2$  decreases from infinity to unity, the polarization changes from 100% in the sense of the maser ( $S_\mu = -I$ ) toward the polarization orthogonal (on the Poincaré sphere) to both the maser and the natural modes ( $S_\perp \rightarrow I$ ).

An interesting case is when the two polarizations (of the maser and medium) are both linear and are perpendicular, in the sense defined here. In terms of polarization vectors, for radiation propagating along the 3-direction, the axes can be chosen such that the two natural modes are polarized along the 1- and 2-axes, respectively, and then the hypothesis that the maser has perpendicular polarization (on the Poincaré sphere) corresponds to its polarization vector being at  $45^\circ$  to either of these. In terms of Stokes parameters, polarization in the sense of the two modes of the medium correspond to  $Q = \pm I$ , respectively, and radiation polarized purely in the sense of the maser corresponds to  $U = I$ . In the foregoing analysis one has  $S_r \rightarrow Q$ ,  $S_\mu \rightarrow U$ , and  $S_\perp \rightarrow V$  in this case. The results imply that for  $\mu^2 > r^2$  the maser produces radiation that is partially circularly polarized, with degree of linear polarization  $U/I = \sqrt{\mu^2 - r^2}/|\mu|$  and degree of circular polarization  $V/I = r/\mu$ . This source of circular polarization in maser sources does not appear to have been recognized previously.

### C. Asymptotic polarization

The analytic results in the general case are cumbersome; they are written down in a concise notation in Appendix B. Of particular interest is the polarization after many growth lengths. The general counterpart of Eq. (18) is derived in Appendix B [cf. Eq. (B4)] by retaining only the terms that

vary as  $\exp(\lambda_+ z)$  in the Mueller matrix (B3). This gives

$$I:S_r:S_\perp:S_\mu = 1: \frac{\lambda_+ \lambda_-}{g \mu r^2} : \frac{\lambda_-^2 + r^2}{\mu r} : - \frac{\lambda_+ (\lambda_-^2 + r^2)}{\mu r^2}. \quad (19)$$

The result (18) is reproduced by Eq. (19) for  $g \rightarrow 0$  with  $\lambda_-^2 \rightarrow -g^2 \mu^2 r^2 / (\mu^2 - r^2)$ ,  $\lambda_+^2 \rightarrow \mu^2 - r^2$ .

An alternative derivation of Eq. (19) is instructive. Suppose that one averages over the oscillations associated with generalized Faraday rotation—for example, assuming that the source extends over a range  $\Delta z \gg 1/\lambda_-$  of  $z$ . Then the real and imaginary parts of the corresponding eigenfunctions are zero. After many growth times, the damping eigenfunction may also be set to zero. The resulting three relations between Stokes parameters implies Eq. (19) for the remaining (growing) eigenfunction. Thus Eq. (19) applies to the growing eigenfunction in a random phase approximation, where the random phase is the generalized Faraday angle, which is  $\lambda_- z$  here. The terms which oscillate [cf. Eq. (B3) in Appendix B] do so with an amplitude that is determined by the initial conditions, so that these oscillations occur with a fixed amplitude on an exponentially growing solution. After many growth lengths the amplitude of the oscillations becomes negligible in comparison with the amplifying component.

### D. Illustrative examples

For strong growth  $\mu^2 \gg r^2$ , one has  $\lambda_+^2 = \mu^2 - r^2(1 - g^2)$ ,  $\lambda_-^2 = -r^2 g^2$ , and Eq. (19) gives  $I:S_r:S_\perp:S_\mu = 1:-g:(r/\mu)(1 - g^2):- (1 - g^2)$ . This corresponds to polarization along the direction  $\mu$  characteristic of the maser, with an admixture  $r/\mu$  of the polarization orthogonal to both the maser and medium. For weak growth  $\mu^2 \ll r^2$ , one has  $\lambda_+^2 = \mu^2 g^2$ ,  $\lambda_-^2 = -r^2 + \mu^2(1 - g^2)$ , and Eq. (19) implies  $I:S_r:S_\perp:S_\mu = 1:- (g/|g|)[1 - (\mu^2/r^2)(1 - g^2)]: (\mu/r) \times (1 - g^2) : - (\mu^2/r^2)|g|(1 - g^2)$ , which corresponds to polarization along the direction of the faster growing of the natural modes, with an admixture  $(\mu/r)(1 - g^2)$  of the polarization orthogonal to both the maser and medium. In the intermediate case where the two rates are equal,  $\mu^2 = r^2$ , Eq. (19) gives  $1:-g^2:1-g^2:-g(1-g^2)$ . This reproduces the special cases of parallel ( $g=1$ ) and perpendicular ( $g=0$ ) polarizations discussed above. In the general case ( $0 < g^2 < 1$ ) all of  $S_r, S_\perp, S_\mu$ , and hence all of  $Q, U, V$  are nonzero in the growing radiation.

In Fig. 1 we show two examples of strong growth and in Fig. 2 two examples of weak growth. In both cases the maser is assumed to be circularly polarized and the natural modes are assumed to be linearly polarized. (For illustrative purposes, the relative signs are chosen such that saturation occurs at or near 1, rather than  $-1$ , in all cases.) However, the plots are generic in the sense that with appropriate relabeling of the polarizations, they apply to any case that has the same value for the ratio  $\mu/r$  and  $g$ . In both figures the plot on the left is for nearly perpendicular polarizations,  $g = -0.1$ , and the plot on the right is for nearly parallel polarizations,  $g = -0.955$ . Note that the only case where the dominant polarization is nearly that of the maser is for strong growth with nearly perpendicular polarizations. In the other cases illus-

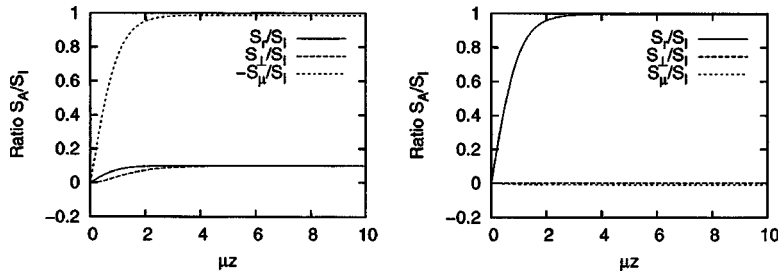


FIG. 1. The Stokes parameters are plotted in two cases where the growth is strong compared with the rate of generalized Faraday rotation. Left: nearly perpendicular case  $\mu_V:r_Q:r_V=-1:0.1:0.01$ . Right: nearly parallel case  $\mu_V:r_Q:r_V=-1:0.01:0.1$ .

trated, the dominant polarization is similar to that of the natural modes of the medium.

A general conclusion is that a maser operating in a medium with natural modes polarized differently to the maser leads to amplifying radiation with polarization that is a mixture of all three Stokes parameters  $Q, U, V$ . The particular ratio is determined by Eq. (19) after a sufficiently large number of growth lengths.

#### E. Growth-induced circular polarization

As noted above, in the case where the maser and natural modes are linearly polarized and are perpendicular in the sense used here, the growing radiation is partially circularly polarized. In this case, “perpendicular” on the Poincaré sphere implies polarization vectors at an angle of  $45^\circ$  to each other. When the two linear polarization vectors are at an angle different from  $45^\circ$ , this effect still occurs. The degree of circular polarization is determined by Eq. (19), with  $S_\perp$  corresponding to circular in this case.

We show three examples that illustrate this case in Fig. 3. The calculations are performed for a circularly polarized maser and linearly polarized modes, but the results are valid for any case with the values of  $\mu/r$  and  $g$  equal to the values chosen in these three cases. Specifically, the three cases correspond to  $\mu/r=-1.005, -1.41, 1.005$  and  $g=-.995, -0.707, 0.100$ , respectively. These three values of  $g$  correspond to nearly parallel, intermediate, and nearly perpendicular vectors on the Poincaré sphere. Ignoring the signs, for linearly polarized maser and natural modes, these three values of  $g$  correspond to angles (modulo  $45^\circ$ )  $9^\circ, 23^\circ$ , and  $42^\circ$ , respectively, between the two polarization vectors. The dashed curves in Fig. 3 correspond to the circular polarization in this case. As expected, the circular polarization is largest for polarizations that are nearly perpendicular on the Poincaré sphere (polarization vectors at  $45^\circ$ ). The degree of circular polarization after many growth lengths approaches the value determined by Eq. (19); specifically, it approaches  $-\lambda_-^2 = r^2/\mu r$ . The sense of circular polarization is determined

by the angle between the linear polarizations of the maser and medium. This is implicit in our description through the sign of  $S_\perp$  which is determined by the direction of  $\hat{\mu} \times \hat{r}$  [cf. Eq. (12)]; this direction—and hence the sense of circular polarization of the growing radiation—changes sign when the angle between the vectors passes through zero (parallel case).

#### IV. DISCUSSION

The main qualitative result from this investigation is that when a completely polarized maser operates in a medium whose natural modes have a different polarization to the maser, the polarization of the amplifying radiation is different from both those of the maser and medium and includes a component orthogonal (on the Poincaré sphere) to both polarizations. This resulting polarization depends on the ratio  $\mu/r$  of the growth rate to the Faraday rotation rate and the cosine of the angle between the two polarization directions (on the Poincaré sphere). The polarization of the growing waves changes from that characteristic of the maser for  $\mu^2 \gg r^2$  to that of the faster-growing natural mode for  $\mu^2 \ll r^2$ .

These results are derived for a completely polarized maser, which is defined to be one with  $\mu_I=0$  in Eqs. (2). This assumption is somewhat artificial and is made to simplify the analysis, but it does not affect the qualitative conclusions. Inclusion of  $\mu_I \neq 0$  leads to an additional factor  $\exp(-\mu_I z)$ , with  $\mu_I < 0$  for a maser, which is common to all Stokes parameters. Hence, the ratios of the Stokes parameters are unchanged from the case of a completely polarized maser. Another simplifying assumption made in this paper is the neglect of spontaneous emission. The polarization of the amplifying radiation becomes independent of the initial radiation after many growth lengths, and the neglect of spontaneous emission does not affect our results.

An interesting result is that when the growth rate and the Faraday rotation rate are comparable, the growing radiation has a component that is orthogonal (on the Poincaré sphere) to both. In particular a linearly polarized maser in a medium

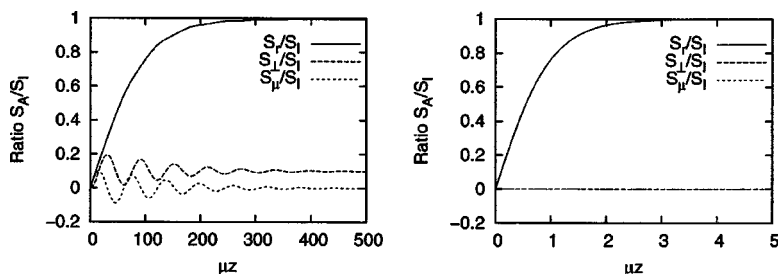


FIG. 2. As for Fig. 1, except for weak growth. Left: nearly perpendicular case  $\mu_V:r_Q:r_V=-0.1:1:0.01$ . Right: nearly parallel case  $\mu_V:r_Q:r_V=-0.1:0.01:1$ .

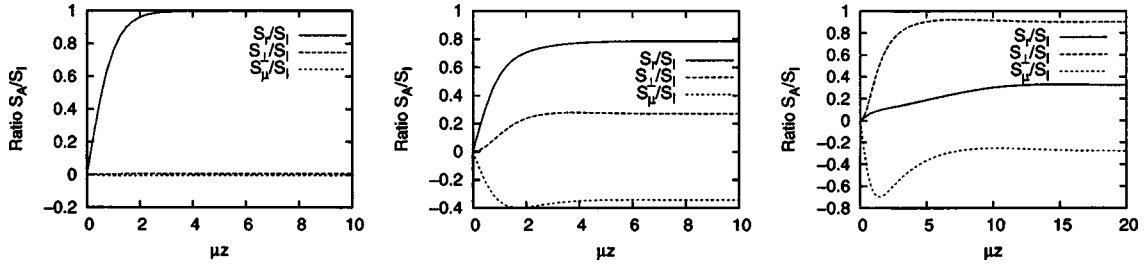


FIG. 3. Left:  $\mu_V = -1$ ,  $r_V = 1$ ,  $r_Q = 0.1$ . Middle:  $\mu_V = -1$ ,  $r_V = 0.5$ ,  $r_Q = 0.5$ . Right:  $\mu_V = -1$ ,  $r_V = 0.1$ ,  $r_Q = 1$ .

with linearly polarized modes that are different from those of the maser leads to growing radiation that can be significantly circularly polarized for  $|\mu/r| \sim 1$ . This is of interest as a possible explanation for the circular polarization observed in pulsar radio emission [12]. The results of this paper may also be relevant to the polarization of interstellar molecular line masers [13] and to the electron cyclotron maser emission, notably in Jupiter's  $S$  bursts where the ellipticity of the radiation can be measured directly [14]. We propose to discuss the suggested application to pulsars in detail elsewhere.

#### ACKNOWLEDGMENT

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#### APPENDIX A: THE MUELLER MATRIX

The Mueller matrix in the present context is defined by Eq. (14). In this appendix direct evaluation of the matrix is discussed in the general case and carried out explicitly in the cases of parallel and perpendicular polarizations.

For  $\mu_I = 0$ , the Mueller matrix (14) is

$$M_{AB}(z) = \exp[s_{AB}z] = \delta_{AB} + s_{AB}z + \frac{s_{AB}^2 z^2}{2!} + \frac{s_{AB}^3 z^3}{3!} + \frac{s_{AB}^4 z^4}{4!} + \dots, \quad (\text{A1})$$

with  $s_{AB} = \mu_I \delta_{AB} - \mu_{AB} + r_{AB}$ . The square of  $s_{AB}$  is an independent matrix, as is its cube, which may be written

$$s_{AB}^3 = \lambda_0^2 s_{AB} + \mu r g t_{AB}, \quad (\text{A2})$$

where the invariants  $\lambda_0$ ,  $g$  are defined by Eqs. (6) and where  $t_{AB}$  is the dual of  $s_{AB}$ . The fourth power of  $s_{AB}$  follows from the characteristic equation (5) and the fact (the Cayley-Hamilton theorem) that a matrix satisfies its characteristic equation, which implies

$$s_{AB}^4 - \lambda_0^2 s_{AB}^2 - \mu^2 r^2 g^2 \delta_{AB} = 0. \quad (\text{A3})$$

The result (A3) also follows by (matrix) multiplying Eq. (A2) by  $s_{AB}$  and using

$$s_{AB} t_{BC} = \mu r g \delta_{AC}. \quad (\text{A4})$$

In this way, using Eqs. (A2) and (A4) or Eq. (A3), all powers of  $s_{AB}$  higher than the second may be reexpressed in terms of the three matrices  $s_{AB}$ ,  $s_{AB}^2$ , and  $t_{AB}$ . However, it is not straightforward to sum the series (A1) except in special cases

including the parallel and perpendicular cases.

For parallel polarizations  $M_{AB}(z)$  factorizes:

$$M_{AB}(z) = m_{AC}(z) R_{BC}(z), \quad m_{AB}(z) = \exp[-\mu_{AB}z],$$

$$R_{AB}(z) = \exp[r_{AB}z]. \quad (\text{A5})$$

Explicit evaluation gives

$$m_{AB}(z) = \delta_{AB} - \mu^{-1} \mu_{AB} \sinh(\mu z) + \mu^{-2} \mu_{AB}^2 [\cosh(\mu z) - 1],$$

$$R_{AB}(z) = \delta_{AB} + r^{-1} r_{AB} \sin(rz) + r^{-2} r_{AB}^2 [\cos(rz) - 1]. \quad (\text{A6})$$

For perpendicular polarizations, one has  $s_{AB}^n = \lambda_0^2 s_{AB}^{n-2}$  for  $n \geq 3$  and the sum of the infinite series gives

$$M_{AB}(z) = \delta_{AB} + s_{AB} \frac{\sinh(\lambda_0 z)}{\lambda_0} + s_{AB}^2 \frac{\cosh(\lambda_0 z) - 1}{\lambda_0^2}, \quad (\text{A7})$$

which applies for  $\lambda_0^2 > 0$ . For  $\lambda_0^2 = -\tilde{\lambda}_0^2 < 0$  the hyperbolic functions are replaced by the corresponding trigonometric functions.

In the case  $\lambda_0^2 = 0$ , where the growth rate and the rate of generalized Faraday rotation are equal, Eq. (A7) reduces to

$$M_{AB}(z) = \delta_{AB} + s_{AB}z + \frac{1}{2} s_{AB}^2 z^2, \quad (\text{A8})$$

and  $s_{AB}$ ,  $s_{AB}^2$  simplify due to  $\mu_V = r_Q$ . This corresponds to power-law rather than exponential growth or damping.

#### APPENDIX B: GENERAL EXPRESSION FOR THE MUELLER MATRIX

The solutions (13) for the four eigenfunctions (12) may be combined into terms that evolve as  $C = \cosh(\lambda_+ z)$ ,  $S = \sinh(\lambda_+ z)$ ,  $c = \cos(|\lambda_-|z)$ , and  $s = \sin(|\lambda_-|z)$ . These combinations are written as a square matrix  $L$  times the Stokes vector in the form  $I, S_r, S_\perp, S_\mu$ . Then the right-hand side of Eq. (13) may be written as another square matrix  $T(z)$  (involving only  $C, S, c, s$ ) times  $L$  times the initial Stokes vector  $I(0), S_r(0), S_\perp(0), S_\mu(0)$ . These square matrices are

$$L = \begin{pmatrix} L_{11} & 0 & L_{13} & 0 \\ 0 & L_{22} & 0 & L_{24} \\ L_{31} & 0 & L_{33} & 0 \\ 0 & L_{42} & 0 & L_{44} \end{pmatrix}, \quad T(z) = \begin{pmatrix} C & S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{pmatrix}, \quad (\text{B1})$$

with  $L_{11} = -\lambda_+(\lambda_+^2 + r^2)$ ,  $L_{13} = \lambda_+\mu r$ ,  $L_{22} = \mu g(\lambda_+^2 + r^2)$ ,  $L_{24} = \mu\lambda_+^2$ ,  $L_{31} = -|\lambda_-|(\lambda_-^2 + r^2)$ ,  $L_{33} = |\lambda_-|\mu r$ , and  $L_{42} = \mu g(\lambda_-^2 + r^2)$ ,

$$L_{44} = \mu\lambda_-^2. \text{ The Mueller matrix is then } M = L^{-1}T(z)L. \text{ With}$$

$$L^{-1} = \frac{1}{D_1 D_2} \begin{pmatrix} D_2 L_{33} & 0 & -D_2 L_{13} & 0 \\ 0 & D_1 L_{44} & 0 & -D_1 L_{24} \\ -D_2 L_{31} & 0 & D_2 L_{11} & 0 \\ 0 & -D_1 L_{42} & 0 & D_1 L_{22} \end{pmatrix}, \quad (\text{B2})$$

$D_1 = L_{11}L_{33} - L_{13}L_{31}$ , and  $D_2 = L_{22}L_{44} - L_{24}L_{42}$ , one has

$$M = \frac{1}{D_1 D_2} \begin{pmatrix} D_2[L_{11}L_{33}C - L_{13}L_{31}c] & D_2[L_{22}L_{33}S - L_{13}L_{42}S] & D_2[L_{13}L_{33}C - L_{13}L_{33}c] & D_2[L_{24}L_{33}S - L_{13}L_{44}S] \\ D_1[L_{11}L_{44}S + L_{24}L_{31}S] & D_1[L_{22}L_{44}C - L_{24}L_{42}C] & D_1[L_{13}L_{44}S + L_{24}L_{33}S] & D_1[L_{24}L_{44}C - L_{24}L_{44}c] \\ D_2[-L_{11}L_{31}C + L_{11}L_{31}c] & D_2[-L_{22}L_{31}S + L_{11}L_{42}S] & D_2[-L_{13}L_{31}C + L_{11}L_{33}c] & D_2[-L_{24}L_{31}S + L_{11}L_{44}S] \\ D_1[-L_{42}L_{11}S - L_{22}L_{31}S] & D_1[-L_{42}L_{22}C + L_{22}L_{42}c] & D_1[-L_{42}L_{13}S - L_{22}L_{33}S] & D_1[-L_{42}L_{24}C + L_{22}L_{44}c] \end{pmatrix}. \quad (\text{B3})$$

After many growth lengths the trigonometric terms are negligible, and Eq. (B3) implies

$$I:S_r:S_\perp:S_\mu = 1:\frac{D_1 L_{44}}{D_2 L_{33}}:-\frac{L_{31}}{L_{33}}:-\frac{D_1 L_{42}}{D_2 L_{33}}, \quad (\text{B4})$$

with the initial polarization appearing only in the combination  $L_{11}I(0) + L_{22}S_r(0) + L_{13}S_\perp(0) + L_{24}S_\mu(0)$ .

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